

Alternative approach to the optimality of the threshold strategy for spectrally negative Lévy processes

^aChuancun Yin*, ^aYing Shen ^bKam Chuen Yuen,

^a*School of Mathematical Sciences, Qufu Normal University, Shandong 273165, China*

^b*Department of Statistics and Actuarial Science, The University of Hong Kong, Pokfulam Road, Hong Kong*

Abstract Consider the optimal dividend problem for an insurance company whose uncontrolled surplus process evolves as a spectrally negative Lévy process. We assume that dividends are paid to the shareholders according to admissible strategies whose dividend rate is bounded by a constant. The objective is to find a dividend policy so as to maximize the expected discounted value of dividends which are paid to the shareholders until the company is ruined. Kyprianou, Loeffen and Pérez [28] have shown that a refraction strategy (also called threshold strategy) forms an optimal strategy under the condition that the Lévy measure has a completely monotone density. In this paper, we propose an alternative approach to this optimal problem.

MSC: 60J51; 93E20; 91B30

Keywords: Spectrally negative Lévy process; Optimal dividend problem; Scale function; Complete monotonicity; Threshold strategy

1 Introduction

The classical optimal dividend problem looks for the strategy that maximizes the expected discounted dividend payments until ruin in an insurance portfolio, which has recently received a lot of attention in actuarial mathematics. This optimization problem was first

*Corresponding author. Tel:+865374453221; fax:+865374455076

E-mail addresses: ccyin@mail.qfnu.edu.cn (C. C. Yin), keyuen@hku.hk (K. C. Yuen)

proposed by De Finetti [11] to reflect more realistically the surplus cash flows in an insurance portfolio, who considered a discrete time random walk with step sizes ± 1 and proved that the optimal dividend strategy is a barrier strategy. Since then many researchers have tried to address this optimality question under more general and more realistic model assumptions and until nowadays this turns out to be a rich and challenging field of research that needs the combination of tools from analysis, probability and stochastic control. For the classical compound Poisson risk model, this problem was solved by Gerber in [15] via a limit of an associated discrete problem. Recently, this optimal dividend problem in the classical compound Poisson risk model and also included a general reinsurance strategy as a second control possibility was taken up again by Azcue and Muler [7], who used stochastic control theory and viscosity solutions. For all these cases in general a band strategy turns out to be optimal among all admissible strategies. In particular, for exponentially distributed claim sizes this optimal strategy simplifies to a barrier strategy. In Albrecher and Thonhauser [1] it is shown that the optimality of barrier strategies in the classical model with exponential claims still holds if there is a constant force of interest. Avram et al. [5] considered the case where the risk process is given by a general spectrally negative Lévy process and gave a sufficient condition involving the generator of the Lévy process for optimality of the barrier strategy. Recently, Loeffen [29] showed that barrier strategy is optimal among all admissible strategies for general spectrally negative Lévy risk processes with completely monotone jump density, and Kyprianou et al. [26] relaxed this condition on the jump density to log-convexity. More recent paper Azcue and Muler [8] examines the analogous questions in the compound Poisson risk model with investment. The corresponding problem in the case of a diffusion risk process was completely solved by Shreve et al. [32] and a barrier strategy was identified to be optimal. The special case of constant drift and diffusion coefficient was then solved again by slightly different means in Jeanblanc-Picqué and Shiryaev [21] and Asmussen and Taksar [4].

Band strategy (and barrier strategy) often serve as candidates for the optimal strategy when the dividend rate is unrestricted. However, the resulting dividend stream is far from practical application. In many circumstances this is not desirable. Furthermore, if a band strategy is applied, ultimate ruin of the company is certain. Motivated by this fact, other dividend strategies such as threshold strategies, linear and nonlinear barrier strategies and multi-layer strategies have been studied. Asmussen and Taksar [4] postulated a bounded dividend rate and showed that the optimal dividend strategy is a threshold strategy in Brownian motion risk models, that is, dividends should be paid out at the maximal admissible rate as soon as the surplus exceeds a certain threshold. Some calculations for this model can be found in [16]. In the compound Poisson risk model, Gerber and Shiu [17] showed that the optimal dividend strategy is a bang bang strategy. In particular, for exponentially distributed claim sizes this optimal strategy simplifies to a threshold strategy. Motivated by Gerber and Shiu [17], Fang and Wu [13] study the analogous questions in the compound Poisson risk model with constant interest, for the case of an exponential claim amount distribution, it is shown that the optimal dividend strategy is a threshold strategy. More recently, Fang and Wu [14] examine the same problem for the Brownian motion risk model with interest. In a very recent paper, Kyprianou, Loeffen and Pérez [28] have shown that a refraction strategy (also called threshold strategy) forms an optimal strategy under the condition that the Lévy measure has a completely monotone density. See Albrecher and Thonhauser [2], Avanzi [6] and Schmidli [31] for nice surveys on this subject. The purpose of this paper is to re-examine the analogous questions in a general spectrally negative Lévy process risk model.

The rest of the paper is organized as follows. In Section 2, we state the problem and

recall some preliminaries on spectrally negative Lévy processes. In Section 3, we will show that the optimal value function of the dividends can be characterized by the Hamilton-Jacobi-Bellman (HJB) equation and give a verification result for optimality. In Section 4 we discuss the threshold strategies. Explicit expressions and the integro-differential equations for the expected discounted value of dividend payments are obtained, and in Section 5 we present the main results, we will show that a threshold strategy forms an optimal strategy under the condition that the Lévy measure has a completely monotone density.

2 Problem setting

Suppose that $X = (X(t) : t \geq 0)$ is a spectrally negative Lévy process with probabilities $\{P_x : x \in \mathbb{R}\}$ such that $X(0) = x$ with probability one, where we write $P = P_0$. Let E_x be the expectation with respect to P_x and write $E = E_0$. Let $\{\mathcal{F}_t : t \geq 0\}$ be the natural filtration satisfying the usual assumptions. Since the jumps of a spectrally negative Lévy process are all non-positive, for convenience, we choose the Lévy measure to have mass only on the positive instead of the negative half line. The Laplace exponent of X is given by

$$\psi(\theta) = a\theta + \frac{1}{2}\sigma^2\theta^2 - \int_0^\infty (1 - e^{-\theta x} - \theta x \mathbf{1}_{\{0 < x < 1\}})\Pi(dx) \quad (2.1)$$

where $a \in \mathbb{R}$, $\sigma \geq 0$ and Π is a measure on $(0, \infty)$ satisfying $\int_0^\infty (1 \wedge x^2)\Pi(dx) < \infty$ and is called the Lévy measure. The characteristics (a, σ^2, Π) are called the Lévy triplet of the process and completely determines its law. ψ is strictly convex on $(0, \infty)$ and satisfies $\psi(0+) = 0$, $\psi(\infty) = \infty$ and $\psi'(0+) = EX(1)$. If $\sigma^2 > 0$ and $\Pi = 0$, then the process is a Brownian motion; When $\sigma^2 = 0$ and $\int_0^\infty \Pi(dx) < \infty$, the process is a compound Poisson process; When $\sigma^2 = 0$, $\int_0^\infty \Pi(dx) = \infty$ and $\int_0^\infty (1 \wedge x)\Pi(dx) < \infty$, the process has an infinite number of small jumps but is of finite variation; When $\sigma^2 = 0$, $\int_0^\infty \Pi(dx) = \infty$ and $\int_0^\infty (1 \wedge x)\Pi(dx) = \infty$, the process has infinitely many jumps and is of unbounded variation. In a word, such a Lévy process has bounded variation if and only if $\sigma = 0$ and $\int_0^1 x\Pi(dx) < \infty$. In this case the Lévy exponent can be re-expressed as

$$\psi(\beta) = c\beta - \int_0^\infty (1 - e^{-\beta x})\Pi(dx),$$

where $c = a + \int_0^1 x\Pi(dx)$ is known as the drift coefficient. If $\sigma^2 > 0$, X is said to have a Gaussian component.

In this paper, we shall only consider the case that Π is absolutely continuous with respect to Lebesgue measure, in which case we shall refer to its density as π .

We recall from Kyprianou [24] that for each $q \geq 0$ there exists a continuous and increasing function $W^{(q)} : \mathbb{R} \rightarrow [0, \infty)$, called the q -scale function defined in such a way that $W^{(q)}(x) = 0$ for all $x < 0$ and on $[0, \infty)$ its Laplace transform is given by

$$\int_0^\infty e^{-\theta x} W^{(q)}(x) dx = \frac{1}{\psi(\theta) - q}, \quad \theta > \Phi(q), \quad (2.2)$$

where $\Phi(q) = \sup\{\theta \geq 0 : \psi(\theta) = q\}$ is the right-inverse of ψ . We shall write W in place of $W^{(0)}$ and call this the scale function rather than the 0-scale function.

The q -scale function takes its name from the identity

$$E_x(e^{-q\tau_a^+} \mathbf{1}(\tau_a^+ < \tau_0^-)) = \frac{W^{(q)}(x)}{W^{(q)}(a)},$$

where

$$\tau_a^+ = \inf\{t \geq 0 : X(t) > a\}, \quad \tau_0^- = \inf\{t \geq 0 : X(t) < 0\}.$$

The following facts about the scale functions are taken from [10, 26]. If X has paths of bounded variation then, for all $q \geq 0$, $W^{(q)}|_{(0,\infty)} \in C^1(0,\infty)$ if and only if Π has no atoms. In the case that X has paths of unbounded variation, it is known that, for all $q \geq 0$, $W^{(q)}|_{(0,\infty)} \in C^1(0,\infty)$. Moreover if $\sigma > 0$ then $C^1(0,\infty)$ may be replaced by $C^2(0,\infty)$. Further, if the Lévy measure has a density, then the scale functions are always differentiable. In particular, if π is completely monotone then $W^{(q)}|_{(0,\infty)} \in C^\infty(0,\infty)$. It is well known that $W^{(\delta)}(0+) = 1/c$ when X has paths of bounded variation. Otherwise $W^{(\delta)}(0+) = 0$ for the case of unbounded variation. In all cases, if $EX(1) > 0$, then $W(\infty) = 1/EX(1)$. If $q > 0$, then $W^{(q)}(x) \sim e^{\Phi(q)x}/\psi'(\Phi(q))$ as $x \rightarrow \infty$.

Spectrally negative Lévy processes have been considered recently in [12, 18, 20, 23, 25, 34], among others, in the context of insurance risk models. It is assumed that, in the absence of dividends, the surplus of a company at time t is $X(t)$. We assume now that the company pay dividends to its shareholders according to some strategy. Let $\xi = \{L_t^\xi : t \geq 0\}$ be a dividend strategy consisting of a left-continuous non-negative non-decreasing process adapted to the filtration $\{\mathcal{F}_t : t \geq 0\}$ of X . L_t^ξ represents the cumulative dividends paid out up to time t under the control ξ by the insurance company whose risk process is modelled by X . We define the controlled risk process $U^\xi = \{U^\xi(t) : t \geq 0\}$ by $U^\xi(t) = X(t) - L_t^\xi$. Let $T = \inf\{t > 0 : U^\xi(t) < 0\}$ be the ruin time, and define the value function of a dividend strategy ξ by

$$V_\xi(x) = E_x \left(\int_0^T e^{-qt} dL_t^\xi \right),$$

where $q > 0$ is the discounted rate.

A dividend strategy is called admissible if $L_{t+}^\xi - L_t^\xi \leq U^\xi(t) \vee 0$ for $t < T$, in other words the lump sum dividend payment is smaller than the size of the available capitals. Let Ξ be the set of all admissible dividend policies. The control problem consists of solving the following stochastic control problem:

$$V_*(x) = \sup_{\xi \in \Xi} V_\xi(x),$$

and, if it exists, to find a strategy $\xi^* \in \Xi$ such that $V_{\xi^*}(x) = V_*(x)$ for all $x \geq 0$.

In this paper, we assume that the admissible dividend rate is $r(t)$ at time t which is bounded by a constant α . In the sequels, we assume that $0 < \alpha < a + \int_0^1 x\pi(x)dx$ if X has paths of bounded variation. Under this additional constraint, we will show that if the Lévy measure Π has a completely monotone density and that $\delta > 0$, then the optimal dividend strategy is formed by a threshold strategy.

3 The HJB equation and verification of optimality

Let

$$V(x) = \sup_{r(\cdot)} V_r(x),$$

where the supremum is taken over all control process $r(t)$ which are admissible according to the constraints. V_r is the value function when the admissible dividend rate is $r(t)$ at time t and V is called the optimal value function. Suppose V is twice continuously differential on $(0, \infty)$ when the process X is of unbounded variation and is continuously differential on $(0, \infty)$ when the process X is of bounded variation. Standard Markovian arguments yield that V satisfies the following Hamilton-Jacobi-Bellman (HJB) equation (the proof is similar to the one in Azcue and Muler [8]):

$$\max_{0 \leq r \leq \alpha} [1 - V'(x)]r + \Gamma V(x, b) - \delta V(x, b) = 0, \quad x \geq 0, \quad (3.1)$$

where

$$\begin{aligned} \Gamma V(x, b) &= \frac{1}{2} \sigma^2 V''(x, b) + a V'(x, b) \\ &\quad + \int_0^\infty [V(x - y, b) - V(x, b) + V'(x, b)y \mathbf{1}_{(0 < y < 1)}] \pi(y) dy. \end{aligned} \quad (3.2)$$

From above expression, as in Gerber and Shiu [17], we see that at time $t \in (0, T)$, the optimal dividend rate is

$$\begin{aligned} r &= 0 \quad \text{if } V'(U^\xi(t-)) > 1, \\ r &= \alpha \quad \text{if } V'(U^\xi(t-)) < 1. \end{aligned}$$

If $V'(U^\xi(t-)) = 1$, the dividend rate r can be any value between 0 and α . In particular, the optimal dividend rate at time 0 is

$$\begin{aligned} r &= 0 \quad \text{if } V'(x) > 1, \\ r &= \alpha \quad \text{if } V'(x) < 1. \end{aligned}$$

Thus, the company should either pay nothing, or the maximum possible. This called a bang bang strategy.

Now we show that a strategy ξ with $\nu(x) \equiv V_\xi(x)$ is smooth enough and satisfying the HJB equation (3.1) is indeed an optimal strategy.

Consider any other dividend strategy, with dividend rate $r(t)$ and surplus $\tilde{X}(t)$ at time t . We claim that

$$E \left(\int_0^T e^{-\delta t} r(t) dt \mid \tilde{X}(0) = x \right) \leq \nu(x). \quad (3.3)$$

From this, it follows that $\nu(x) = V(x)$, and hence the given strategy ξ is optimal.

To prove inequality (3.3), we consider the martingale

$$e^{-\delta t} \nu(\tilde{X}(t)) - \int_0^t e^{-\delta s} \left((\Gamma - \delta) \nu(\tilde{X}(s)) - r(s) \nu'(\tilde{X}(s)) \right) ds,$$

which can be shown by Itô's formula for semimartingale. From optimal sampling theorem, we have

$$E \left(e^{-\delta(t \wedge T)} \nu(\tilde{X}(t \wedge T)) - \int_0^{t \wedge T} e^{-\delta s} \left((\Gamma - \delta) \nu(\tilde{X}(s)) - r(s) \nu'(\tilde{X}(s)) \right) ds \middle| \tilde{X}(0) = x \right) = \nu(x),$$

which implies

$$-E \left(\int_0^{t \wedge T} e^{-\delta s} \left((\Gamma - \delta) \nu(\tilde{X}(s)) - r(s) \nu'(\tilde{X}(s)) \right) ds \middle| \tilde{X}(0) = x \right) \leq \nu(x),$$

since

$$E \left(e^{-\delta(t \wedge T)} \nu(\tilde{X}(t \wedge T)) \middle| \tilde{X}(0) = x \right) \geq 0.$$

Because the function $\nu(x)$ satisfies the HJB equation (3.1), we have

$$r(s) + (\Gamma - \delta) \nu(\tilde{X}(s)) - r(s) \nu'(\tilde{X}(s)) \leq 0.$$

Thus

$$\begin{aligned} E \left(\int_0^{t \wedge T} e^{-\delta s} r(s) ds \middle| \tilde{X}(0) = x \right) \\ \leq -E \left(\int_0^{t \wedge T} e^{-\delta s} \left((\Gamma - \delta) \nu(\tilde{X}(s)) - r(s) \nu'(\tilde{X}(s)) \right) ds \middle| \tilde{X}(0) = x \right) \leq \nu(x). \end{aligned}$$

Letting $t \rightarrow \infty$ yields (3.3).

4 Threshold dividend strategies

In this section, we study the threshold strategy. We assume that the company pays dividends according to the following strategy governed by parameters $b > 0$ and $\alpha > 0$. Whenever the modified surplus is below the threshold level b , no dividends are paid. However, when the surplus is above this threshold level, dividends are paid continuously at a constant rate α that does not exceed the premium rate c . Note that if $\alpha = c$, we have a barrier strategy again. We define the modified risk process $U_b = \{U_b(t) : t \geq 0\}$ by $U_b(t) = X(t) - D_b(t)$, where $D_b(t) = \alpha \int_0^t \mathbf{1}(U_b(s) > b) ds$. The existence of such a process and some conclusions on fluctuation identities can be found in Kyprianou and Loeffen [27]. Let D_b denote the present value of all dividends until time of ruin T ,

$$D_b = \alpha \int_0^T e^{-\delta t} \mathbf{1}(U_b(t) > b) dt$$

where $T = \inf\{t > 0 : U_b(t) < 0\}$ with $T = \infty$ if $U_b(t) \geq 0$ for all $t \geq 0$. Here $\delta > 0$ is the discount factor. Denote by $V(x, b)$ the expected discounted value of dividend payments, that is,

$$V(x, b) = E(D_b | U_b(0) = x).$$

Clearly, $0 \leq V(x, b) \leq \frac{\alpha}{\delta}$ and $\lim_{x \rightarrow \infty} V(x, b) = \frac{\alpha}{\delta}$.

Define the first passage times, with the convention $\inf \emptyset = \infty$,

$$T_b^+ = \inf\{t \geq 0 : U_b(t) > b\}, \quad T_b^- = \inf\{t \geq 0 : U_b(t) \leq b\}.$$

Let $Y = \{Y(t) := X(t) - \alpha t\}_{t \geq 0}$. For each $\delta \geq 0$, $W^{(\delta)}$ and $Z^{(\delta)}$ are the δ -scale functions associated with X and $W_*^{(\delta)}$ and $Z_*^{(\delta)}$ are the δ -scale functions associated with Y . Further, Ψ is defined as the right inverse of the Laplace exponent of Y so that

$$\Psi(\delta) = \sup\{\theta \geq 0 : \psi(\theta) - \alpha\theta = \delta\}.$$

Theorem 4.1. *Assume $W^{(\delta)}$ is continuously differentiable on $(0, \infty)$.*

(1) *For $0 \leq x \leq b$, we have*

$$V(x, b) = V(b, b) \frac{W^{(\delta)}(x)}{W^{(\delta)}(b)}. \quad (4.1)$$

(2) *For $x > b$, we have*

$$\begin{aligned} V(x, b) &= -\alpha \int_0^{x-b} W_*^{(\delta)}(y) dy + \frac{\alpha}{\Psi(\delta)} W_*^{(\delta)}(x-b) \\ &\quad + \frac{\alpha^2}{2} V(b, b) \left(W_*^{(\delta)'}(x-b) - \Psi(\delta) W_*^{(\delta)}(x-b) \right) \\ &\quad + \frac{V(b, b)}{W^{(\delta)}(b)} \int_b^\infty \int_{-\infty}^b u_*^{(\delta)}(x-b, y-b) \pi(y-z) W^{(\delta)}(z) dy dz. \end{aligned} \quad (4.2)$$

where

$$u_*^{(\delta)}(x, y) = W_*^{(\delta)}(x) e^{-\Psi(\delta)y} - W_*^{(\delta)}(x-y).$$

Proof (1). For $0 \leq x \leq b$, using the strong Markov property of U at T_b^+ , we have

$$V(x, b) = V(b, b) E_x(e^{-\delta T_b^+} \mathbf{1}(T_b^+ < T)),$$

and (4.1) follows since

$$E_x(e^{-\delta T_b^+} \mathbf{1}(T_b^+ < T)) = \frac{W^{(\delta)}(x)}{W^{(\delta)}(b)}.$$

(2). For $x > b$, using the strong Markov property of U_b at T_b^- , we have

$$\begin{aligned} V(x, b) &= \frac{\alpha}{\delta} P_x(T_b^- = \infty) + \alpha E_x \left(\int_0^{T_b^-} e^{-\delta u} du, T_b^- < \infty \right) \\ &\quad + \alpha E_x \left(\int_{T_b^-}^T \mathbf{1}(U_b(u) > b) e^{-\delta u} du, T_b^- < \infty \right) \\ &= \frac{\alpha}{\delta} P_x(T_b^- = \infty) + \frac{\alpha}{\delta} E_x \left((1 - e^{-\delta T_b^-}), T_b^- < \infty \right) \\ &\quad + E_x \left(e^{-\delta T_b^-} V(U_b(T_b^-), b), T_b^- < \infty \right) \\ &= \frac{\alpha}{\delta} - \frac{\alpha}{\delta} E_x \left(e^{-\delta T_b^-}, T_b^- < \infty \right) \\ &\quad + E_x \left(e^{-\delta T_b^-} V(U_b(T_b^-), b), T_b^- < \infty \right). \end{aligned} \quad (4.3)$$

Note that

$$\begin{aligned}
E_x \left(e^{-\delta T_b^-} V(U_b(T_b^-), b), T_b^- < \infty \right) &= V(b, b) E_x \left(e^{-\delta T_b^-}, U_b(T_b^-) = b \right) \\
&+ \int_b^\infty \int_{-\infty}^b u_*^{(\delta)}(x-b, y-b) \pi(y-z) V(z, b) dy dz \\
&= \frac{\sigma^2}{2} V(b, b) \left(W_*^{(\delta)'}(x-b) - \Psi(\delta) W_*^{(\delta)}(x-b) \right) \\
&+ \int_b^\infty \int_{-\infty}^b u_*^{(\delta)}(x-b, y-b) \pi(y-z) V(z, b) dy dz.
\end{aligned}$$

In particular,

$$\begin{aligned}
E_x \left(e^{-\delta T_b^-}, T_b^- < \infty \right) &= \frac{\sigma^2}{2} \left(W_*^{(\delta)'}(x-b) - \Psi(\delta) W_*^{(\delta)}(x-b) \right) \\
&+ \int_b^\infty \int_{-\infty}^b u_*^{(\delta)}(x-b, y-b) \pi(y-z) dy dz \\
&= Z_*^{(\delta)}(x-b) - \frac{\delta}{\Psi(\delta)} W_*^{(\delta)}(x-b),
\end{aligned}$$

where

$$Z_*^{(\delta)}(x) = 1 + \delta \int_0^x W_*^{(\delta)}(y) dy.$$

It follows that

$$\begin{aligned}
V(x, b) &= -\alpha \int_0^{x-b} W_*^{(\delta)}(y) dy + \frac{\alpha}{\Psi(\delta)} W_*^{(\delta)}(x-b) \\
&+ \frac{\sigma^2}{2} V(b, b) \left(W_*^{(\delta)'}(x-b) - \Psi(\delta) W_*^{(\delta)}(x-b) \right) \\
&+ \int_b^\infty \int_{-\infty}^b u_*^{(\delta)}(x-b, y-b) \pi(y-z) V(z, b) dy dz.
\end{aligned}$$

Putting (4.1) into above expression leads to (4.2).

The following result agrees with the result of Kyprianou and Loeffen [27, (10.25)].

Corollary 4.1. *Suppose X has paths of bounded variation and let $0 < \alpha < c$, where $c = a + \int_0^1 x \Pi(dx)$.*

(1) *For $0 \leq x \leq b$, we have*

$$V(x, b) = \frac{W^{(\delta)}(x)}{\Psi(\delta) e^{\Psi(\delta)b} \int_b^\infty e^{-\Psi(\delta)z} W^{(\delta)'}(z) dz}. \quad (4.4)$$

(2) *For $x > b$, we have*

$$V(x, b) = -\alpha \int_0^{x-b} W_*^{(\delta)}(y) dy + \frac{W^{(\delta)} + \alpha \int_b^x W_*^{(\delta)}(x-y) W^{(\delta)'}(y) dy}{\Psi(\delta) e^{\Psi(\delta)b} \int_b^\infty e^{-\Psi(\delta)z} W^{(\delta)'}(z) dz}. \quad (4.5)$$

Proof (1). It follows from Kyprianou and Loeffen [27, (4.10)] that, for $x > b$,

$$\begin{aligned} & \int_b^\infty \int_{-\infty}^b u_*^{(\delta)}(x-b, y-b) \pi(y-z) W^{(\delta)}(z) dy dz \\ &= W^{(\delta)}(x) + \alpha \int_b^x W_*^{(\delta)}(x-z) W^{(\delta)'}(z) dz \\ & - \alpha W_*^{(\delta)}(x-b) e^{\Psi(\delta)b} \int_b^\infty e^{-\Psi(\delta)z} W^{(\delta)'}(z) dz. \end{aligned}$$

Substituting this into (4.2) and letting $x \rightarrow b$ we find that

$$V(b, b) = \frac{W^{(\delta)}(b)}{\Psi(\delta) e^{\Psi(\delta)b} \int_b^\infty e^{-\Psi(\delta)z} W^{(\delta)'}(z) dz}.$$

This, together with (4.1) and (4.2), leads to (4.4) and (4.5).

The next result was obtained by Gerber and Shiu [17] for the compound Poisson model:

Theorem 4.2. *Suppose that X has no Gaussian component. Then, as a function of x , $V(x, b)$ satisfies the following integro-differential equations:*

$$\begin{aligned} aV'(x, b) &+ \int_0^\infty [V(x-y, b) - V(x, b) + V'(x, b)y \mathbf{1}_{(0 < y < 1)}] \pi(y) dy \\ &= \delta V(x, b), \quad 0 < x < b, \end{aligned} \tag{4.6}$$

$$\begin{aligned} (a - \alpha)V'(x, b) &+ \int_0^\infty [V(x-y, b) - V(x, b) + V'(x, b)y \mathbf{1}_{(0 < y < 1)}] \pi(y) dy \\ &= \delta V(x, b) - \alpha, \quad x > b, \end{aligned} \tag{4.7}$$

with the continuity condition $V(b-, b) = V(b+, b) = V(b, b)$. Moreover, if X has paths of bounded variation then

$$cV'(b-, b) = (c - \alpha)V'(b+, b) + \alpha;$$

If X has paths of unbounded variation then

$$V'(b-, b) = V'(b+, b),$$

where $c = a + \int_0^1 x\pi(x)dx$.

Proof Equations (4.6) and (4.7) can be proved by Ito's formula. It follows from (4.1) that $V(b-, b) = V(b, b)$, since $W^{(\delta)}$ is continuous at b . From (4.2) we see that $V(x, b)$ is differential on $[b, \infty)$, where we at $x = b$ mean the right-hand derivative. Consequently, $V(b+, b) = V(b, b)$. This proves the continuity of V at b . If X has paths of bounded variation, then the derivative, $V'(x, b)$, is not necessarily continuous at $x = b$. In fact, it follows from equations (4.6) and (4.7) that

$$cV'(b-, b) = (c - \alpha)V'(b+, b) + \alpha.$$

If X has paths of unbounded variation, let Π_n be measures on $(1/n, \infty)$:

$$\Pi_n(dx) = \Pi(dx)\mathbf{1}_{(1/n, \infty)}, \quad n \geq 1.$$

Then $\int_0^1 x \Pi_n(dx) < \infty$. This is to say that a process X_n with the Lévy measure Π_n has paths of bounded variation. From above we get

$$c_n V'_n(b-, b) = (c_n - \alpha) V'_n(b+, b) + \alpha,$$

where $c_n = a + \int_0^1 x \Pi_n(x) dx$. Letting $n \rightarrow \infty$ yields

$$V'(b-, b) = V'(b+, b).$$

This ends the proof of Theorem 4.2.

The next result was obtained by Wan [33] for the compound Poisson model perturbed by diffusion:

Theorem 4.3. *Suppose that X has a Gaussian component $\sigma > 0$. Then, as a function of x , $V(x, b)$ satisfies the following integro-differential equations:*

$$\begin{aligned} \frac{1}{2}\sigma^2 V''(x, b) &+ aV'(x, b) + \int_0^\infty [V(x-y, b) - V(x, b) + V'(x, b)y\mathbf{1}_{(0 < y < 1)}]\pi(y)dy \\ &= \delta V(x, b), \quad 0 < x < b, \end{aligned} \tag{4.8}$$

$$\begin{aligned} \frac{1}{2}\sigma^2 V''(x, b) &+ (a - \alpha)V'(x, b) + \int_0^\infty [V(x-y, b) - V(x, b) + V'(x, b)y\mathbf{1}_{(0 < y < 1)}]\pi(y)dy \\ &= \delta V(x, b) - \alpha, \quad x > b, \end{aligned} \tag{4.9}$$

with the boundary conditions $V(0, b) = 0$. Moreover

$$V(b-, b) = V(b+, b) = V(b, b), \quad V'(b-, b) = V'(b+, b).$$

Proof Equations (4.8) and (4.9) can be proved by Ito's formula. If $U_b(0) = 0$, because $\sigma > 0$, ruin is immediate and no dividend is paid, so we have $V(0, b) = 0$. It follows from (4.1) that $V(b-, b) = V(b, b)$, since $W^{(\delta)}$ is continuous at b . From (4.3) we have

$$V(b+, b) \leq \frac{\alpha}{\delta} \left(1 - E_b e^{-\delta T_b^-}\right) + V(b, b) E_b \left(e^{-\delta T_b^-}\right) = V(b, b),$$

and thus $V(b+, b) = V(b, b)$. This proves the continuity of V at b .

In the following, we prove that $\{\sigma B(t); t \geq 0\}$ can be approximated by the process $\{c_\varepsilon t - \varepsilon N_\varepsilon(t); t \geq 0\}$, where $N_\varepsilon(t)$ is a Poisson process with parameter $\lambda_\varepsilon > 0$, and $c_\varepsilon > 0$ is a constant. Now, we choose ε , λ_ε , and c_ε such that $\text{Var}[\varepsilon N_\varepsilon(t)] = \sigma^2 t$ and $E[c_\varepsilon t - \varepsilon N_\varepsilon(t)] = 0$. These two conditions yield $\lambda_\varepsilon = \sigma^2/\varepsilon^2$ and $c_\varepsilon = \sigma^2/\varepsilon$. It is easy to prove that, when $\varepsilon \rightarrow 0^+$, $E[e^{z(c_\varepsilon t - \varepsilon N_\varepsilon(t))}] \rightarrow e^{z^2 \sigma^2 t/2}$. This shows that the process $\{c_\varepsilon t - \varepsilon N_\varepsilon(t); t \geq 0\}$ converges weakly to the process $\{\sigma B(t); t \geq 0\}$. It follows that the Lévy process X with Lévy triplet (a, σ, Π) can be approximated by the Lévy process X_ε

with Lévy triplet $(a + c_\varepsilon, 0, \Pi_\varepsilon)$, where $\Pi_\varepsilon = \Pi + \mathbf{1}_{(x \geq \varepsilon)}$. Therefore, by Theorem 4.2, for example, in the case of bounded variation, we have

$$(c + c_\varepsilon)V'_\varepsilon(b-, b) = (c + c_\varepsilon - \alpha)V'_\varepsilon(b+, b) + \alpha.$$

Letting $\varepsilon \rightarrow 0$ and noting that $c, \alpha, V'_\varepsilon(b-, b)$ and $V_\varepsilon(b+, b)$ are bounded, and $\lim_{\varepsilon \rightarrow 0} V_\varepsilon = V$, yields

$$V'(b-, b) = V'(b+, b).$$

This ends the proof of Theorem 4.3.

5 Optimal dividend strategies

In some situations the optimal dividend strategy is a threshold strategy. It is easy to see that if $V'(x, 0) < 1$ for $x > 0$, then the threshold strategy with $b^* = 0$ is optimal, if $V'(x, b^*) > 1$ for $x < b^*$ and $V'(x, b^*) < 1$ for $x > b^*$, then the threshold strategy with $b^* > 0$ is optimal. The optimal threshold b^* can be obtained $V'(b^*, b^*) = 1$. In fact, for the case $\sigma > 0$, it is clearly, since $V'(x, b)$ is a continuous functions of x on $(0, \infty)$; For the case $\sigma = 0$, using the the same argument as in Gerber and Shiu [17], the result follows. From those facts one sees that if $V(x, b^*)$ is a continuously differentiable concave function on $(0, \infty)$, then the optimal dividend strategy is a threshold strategy.

We now review definitions and some properties of logconvex functions and completely monotone functions. We refer the readers to [3, 9] for more details.

A function f defined on an convex subset of a real vector space and taking positive values is said to be logarithmically convex if $\log(f(x))$ is a convex function of x . It is easy to see that a logarithmically convex function is a convex function, but the converse is not always true. For example $f(x) = x^2$ is a convex function, but $\log(f(x)) = 2 \log |x|$ is not a convex function and thus $f(x) = x^2$ is not logarithmically convex.

Recall that a $f \in C^\infty(0, \infty)$ with $f \geq 0$ is completely monotone if its derivatives alternate in sign, i.e. $(-1)^n f^{(n)} \geq 0$ for all $n \in \mathbb{N}$.

Note that the class of logconvex functions contains the class of completely monotone functions. In fact, any completely monotone function is both nonincreasing and logconvex.

Some distributions with completely monotone density functions are (see [9, 29]):

- Weibull distribution with density: $f(x) = crx^{r-1}e^{-cx^r}$, $x > 0$, with $c > 0$ and $0 < r < 1$.
- Pareto distribution with density: $f(x) = \alpha(1+x)^{-\alpha-1}$, $x > 0$, with $\alpha > 0$.
- Mixture of exponential densities: $f(x) = \sum_{i=1}^n A_i \beta_i e^{-\beta_i x}$, $x > 0$, with $A_i > 0, \beta_i > 0$ for $i = 1, 2, \dots, n$, and $\sum_{i=1}^n A_i = 1$.

- Gamma distribution with density: $f(x) = \frac{x^{c-1}e^{-x/\beta}}{\Gamma(c)\beta^c}$, $x > 0$, with $\beta > 0$, $0 < c \leq 1$.

The following are several important examples of spectrally negative Lévy processes with completely monotone densities and that satisfy $\int_0^\infty \pi(x)dx = \infty$ (cf. [19, 29]):

- α -stable process with Lévy density: $\pi(x) = \lambda x^{-1-\alpha}$, $x > 0$ with $\lambda > 0$ and $\alpha \in (0, 1) \cup (1, 2)$;
- One-sided tempered stable process (particular cases include gamma process ($\alpha = 0$) and inverse Gaussian process ($\alpha = \frac{1}{2}$)) with Lévy density: $\pi(x) = \lambda x^{-1-\alpha}e^{-\beta x}$, $x > 0$ with $\beta, \lambda > 0$ and $-1 \leq \alpha < 2$;
- The associated parent process with Lévy density: $\pi(x) = \lambda_1 x^{-1-\alpha}e^{-\beta x} + \lambda_2 x^{-2-\alpha}e^{-\beta x}$, $x > 0$ with $\lambda_1, \lambda_2 > 0$ and $-1 \leq \alpha < 1$.

More examples can be found in recent paper of Jeannin and Pistorius [22, Example 2.4].

The following result can be found in Loeffen and Renaud [30]:

Lemma 5.1. *Suppose the tail of the Lévy measure is log-convex, then, for all $\delta \geq 0$, $W^{(\delta)}$ has a log-convex first derivative.*

Theorem 5.1. *Suppose the tail of the Lévy measure is log-convex, then, for all $\delta \geq 0$, $V(x, b^*)$ is a concave function on $(0, b^*)$, where b^* is the solution of $V'(b^*, b^*) = 1$.*

Proof Differentiate (4.1) with respect to x , and then set $x = b^*$ yields

$$V'(x, b^*) = \frac{W^{(\delta)'}(x)}{W^{(\delta)'}(b^*)}, \quad 0 \leq x \leq b^*. \quad (5.1)$$

Since b^* is the value of b that maximizes $V(x, b)$, i.e. b^* is the value where $W^{(\delta)'}(b)$ attains its global minimum, and thus $W^{(\delta)'}(x)$ is decreasing on $(0, b^*)$. It follows that $V'(x, b^*)$ is decreasing on $(0, b^*)$, which implies $V(x, b^*)$ is a concave function on $(0, b^*)$.

Theorem 5.2. *Suppose that the Lévy density π is a completely monotone function on $(0, \infty)$ and that $\delta > 0$. Then $V(x, b^*)$ is a concave function on (b^*, ∞) , where b^* is the solution of $V'(b^*, b^*) = 1$.*

Proof We prove the theorem in three cases.

Case 1: $\Pi(0, \infty) < \infty$ and $\sigma = 0$. Using (4.5) and repeating the proof of Lemma 8 in Kyprianou, Loeffen and Pérez [28] we get $V''(x, b^*) \leq 0$, $x > b^*$.

Case 2: $\Pi(0, \infty) = \infty$ and $\sigma = 0$. By Bernstein's theorem we can express the function π in the form:

$$\pi(x) = \int_0^\infty e^{-ux} \mu(du),$$

where μ is a measure on $(0, \infty)$. It is well known that π can be approximated arbitrarily closely by a hyper-exponential density (by approximating the measure μ by sums of point masses). It follows from Jeannin and Pistorius [22] that X can be approximated by a sequence of approximating processes $(X^{(n)})_{n \geq 1}$. The approximating density equal to

$$\pi_n(x) = \sum_i e^{-xu_i} \Delta_i,$$

where $(u_i)_i = (u_i^{(n)})_i$ is the finite partitions of $(0, \infty)$, and $(\Delta_i)_i = (\Delta_i^{(n)})_i$ is finite sets of positive weights. For a given n , the partitions $(u_i)_i$ and the weights $(\Delta_i)_i$ are satisfy certain conditions. The approximating process $X^{(n)}$ constructed in this way can be shown to converge weakly to X . For more details, see Jeannin and Pistorius [22, §3.2]. For each $n \geq 1$, we find that $\int_0^\infty \pi_n(x) dx < \infty$ and π_n is complete monotone on $(0, \infty)$. Let V_n be the expected discounted value of dividend payments corresponding to the Lévy process $(X^{(n)})_{n \geq 1}$. From the Case 1, we know that $V_n(x, b_n^*)$ is a concave function on (b_n^*, ∞) , where b_n^* is the solution of $V_n'(b_n^*, b_n^*) = 1$. The result follows, since $\lim_{n \rightarrow \infty} V_n = V$, $\lim_{n \rightarrow \infty} V_n' = V'$ and the limit of a pointwise convergent sequence of concave functions is concave.

Case 3: $\sigma \neq 0$. We consider the following approximating process $(X^{(n)})_{n \geq 1}$ (cf. Loeffen and Renaud [30]): The Lévy triplet is $(a_n, 0, \Pi_n)$, where $a_n = a + \frac{1}{2}\sigma^2 n e^{-n}(n+1)$ and, Π_n is defined by

$$\Pi_n(x, \infty) = \Pi(x, \infty) + \frac{1}{2}\sigma^2 n^2 e^{-nx}.$$

For all $\theta \geq 0$, $\lim_{n \rightarrow \infty} \psi_n(\theta) = \psi(\theta)$, and thus, by the continuity theorem for Laplace transforms, $\lim_{n \rightarrow \infty} W_n^{(\delta)}(x) = W^{(\delta)}(x)$ for all $x \geq 0$. Let V_n be the expected discounted value of dividend payments corresponding to the Lévy process $X^{(n)}$. It is easy to see that Π_n has a completely monotone density and thus from the Case 1 and Case 2, we know that $V_n(x, b_n^*)$ is a concave function on (b_n^*, ∞) , where b_n^* is the solution of $V_n'(b_n^*, b_n^*) = 1$. Because $\lim_{n \rightarrow \infty} V_n = V$, $\lim_{n \rightarrow \infty} V_n' = V'$, the result follows. This completes the proof of Theorem 5.2.

Combining Theorems 5.1 and 5.2 we get the main result of this paper which established by Kyprianou, Loeffen and Pérez [28] by using an alternative argument:

Theorem 5.3. *Suppose that the Lévy measure Π has a completely monotone density and that $\delta > 0$. Then $V(x, b^*)$ is a concave function on $(0, \infty)$, where b^* is the solution of $V'(b^*, b^*) = 1$. Consequently, the threshold strategy with threshold b^* is the optimal dividend strategy.*

Remark 5.1. If there were no restrictions on the admissible dividend rate $r(t)$, or, $\alpha = c$ in the case of X has paths of bounded variation, then the threshold becomes a barrier. Suppose that the Lévy measure Π has a completely monotone density and that $\delta > 0$, then barrier strategy at b^* is an optimal strategy. This result can be found in Loeffen [29]. Kyprianou et al. [26] strengthen the result of Loeffen [29] established a larger class of Lévy processes for which the barrier strategy is optimal among all admissible ones. An alternative proof is given in Yin and Wang [35].

Remark 5.2. We can conclude that the threshold strategy is the optimal dividend strategy for the compound Poisson risk model or the compound Poisson risk model perturbed

by Brownian motion where the claims have a distribution with a completely monotone probability density function, which extended the known result (c.f. Gerber and Shiu [17] where only consider the exponential claim density under the Cramér-Lundberg setting).

Acknowledgements The research was supported by the National Natural Science Foundation of China (No. 10771119) and the Research Fund for the Doctoral Program of Higher Education of China (No. 20093705110002).

References

- [1] H. Albrecher, S. Thonhauser, Optimal dividend strategies for a risk process under force of interest, *Insurance: Math. Econ.* 43 (2008) 134-149.
- [2] H. Albrecher, S. Thonhauser, Optimality results for dividend problems in insurance, *Rev. R. Acad. Cien. Serie A. Mat.* 103 (2009) 295-320.
- [3] Y. An, Logconcavity versus logconvexity: A complete characterization, *J. Econom. Theory* 80 (1998) 350-369.
- [4] S. Asmussen, M. Taksar, Controlled diffusion models for optimal dividend pay-out, *Insurance Math. Econom.* 20 (1997) 1-15.
- [5] F. Avram, Z. Palmowski, M. R. Pistorius, On the optimal dividend problem for a spectrally negative Lévy process, *Ann. Appl. Probab.* 17 (2007) 156-180.
- [6] B. Avanzi, Strategies for dividend distribution: A review, *North Amer. Actuar. J.* 13 (2009) 217-249.
- [7] P. Azcue, N. Muler, Optimal reinsurance and dividend distribution policies in the Cramér-Lundberg model, *Math. Finance* 15 (2005) 261-308.
- [8] P. Azcue, N. Muler, Optimal investment policy and dividend payment strategy in an insurance company, *Ann. Appl. Probab.* 20 (4) (2010) 1253-1302.
- [9] M. Bagnoli, T. Bergstrom, Log-concave probability and its applications, *Economic Theory* 26 (2005) 445-469.
- [10] T. Chan, A. E. Kyprianou, M. Savov, Smoothness of scale functions for spectrally negative Lévy processes. DOI: 10.1007/s00440-010-0289-4 (To appear in *Probability Theory and Related Fields*, 2010.)
- [11] B. De Finetti, Su un'impostazione alternativa della teoria collettiva del rischio, *Transactions of the XVth International Congress of Actuaries* 2 (1957) 433-443.
- [12] A. D. Doney, A. E. Kyprianou, Overshoots and undershoots of Lévy processes, *Ann. Appl. Probab.* 16 (2006) 91-106.
- [13] Y. Fang, R. Wu, Optimal dividend strategy in the compound Poisson model with constant interest, *Stochastic Models* 23 (2007) 149-166.

- [14] Y. Fang, R. Wu, Optimal dividends in the Brownian motion risk model with interest, *J. Comput. Appl. Math.* 229 (2009) 145-151.
- [15] H. U. Gerber, Entscheidungskriterien für den zusammengesetzten Poisson-Prozess, *Mitteilungen der Vereinigung Schweizerischer Versicherungsmathematiker* 69 (1969) 185-227.
- [16] H. U. Gerber, E. S. W. Shiu, Optimal dividends: Analysis with Brownian motion, *North Amer. Actuar. J.* 8 (2004) 1-20.
- [17] H. U. Gerber, E. S. W. Shiu, On optimal dividend strategies in the compound Poisson model, *North Amer. Actuar. J.* 10 (2006) 76-93.
- [18] J. Garrido, M. Morales, On the expected discounted penalty function for Lévy risk processes, *North Amer. Actuar. J.* 10 (2006) 196-218.
- [19] F. Hubalek, A. E. Kyprianou, Old and new examples of scale functions for spectrally negative Lévy processes. arXiv:0801.0393v1 [math.PR]. To appear in *Sixth Seminar on Stochastic Analysis, Random Fields and Applications*, eds R. Dalang, M. Dozzi, F. Russo. *Progress in Probability*, Birkhäuser (2010).
- [20] M. M. Huzak, M. Perman, H. Šikić, Z. Vondraček, Ruin probabilities and decompositions for general perturbed risk processes, *Ann. Appl. Probab.* 14 (2004) 1378-1397
- [21] M. Jeanblanc-Picqué, A. N. Shiryaev, Optimization of the flow of dividends, *Russian Math. Surveys* 50 (1995) 257-277.
- [22] M. Jeannin, M. Pistorius, A transform approach to compute prices and Greeks of barrier options driven by a class of Lévy processes, *Quantitative Finance* 10 (6)(2010) 629-644.
- [23] C. Klüppelberg, A. E. Kyprianou, R. A. Maller, Ruin probabilities and overshoots for general Lévy insurance risk processes, *Ann. Appl. Probab.* 14 (2004) 1766-1801.
- [24] A. E. Kyprianou, *Introductory Lectures on Fluctuations of Lévy processes with Applications*, Springer-Verlag, Berlin, 2006.
- [25] A. E. Kyprianou, Z. Palmowski, Distributional study of de Finetti's dividend problem for a general Lévy insurance risk process, *J. Appl. Probab.* 44 (2007) 428-443.
- [26] A. E. Kyprianou, V. Rivero, R. Song, Convexity and smoothness of scale functions with applications to de Finetti's control problem, *Journal of Theoretical Probability* 23 (2010) 547-564.
- [27] A. E. Kyprianou, R. Loeffen, Refracted Lévy processes, *Annales de l'Institut Henri Poincaré-Probabilités et Statistiques* 46 (2010) 24-44
- [28] A. E. Kyprianou, R. Loeffen and J. L. Pérez, Optimal control with absolutely continuous strategies for spectrally negative Lévy processes. Arxiv preprint arXiv:1008.2363, 2010- arxiv.org
- [29] R. Loeffen, On optimality of the barrier strategy in de Finetti's dividend problem for spectrally negative Lévy processes, *Ann. Appl. Probab.* 18 (2008) 1669-1680.

- [30] R. Loeffen, J. Renaud, De Finetti's optimal dividends problem with an affine penalty function at ruin, *Insurance: Mathematics and Economics* 46 (2010) 98-108
- [31] H. Schmidli, *Stochastic Control in Insurance*, Springer, 2008.
- [32] S. E. Shreve, J. P. Lehoczky, D. P. Gaver, Optimal consumption for general diffusions with absorbing and reflecting barriers, *SIAM J. Control Optim.* 22 (1984) 55-75.
- [33] N. Wan, Dividend payments with a threshold strategy in the compound Poisson risk model perturbed by diffusion, *Insurance: Mathematics and Economics* 40 (2007) 509-523.
- [34] H. L. Yang, L. Z. Zhang, Spectrally negative Lévy processes with applications in risk theory, *Adv. Appl. Probab.* 33 (2001) 281-291.
- [35] C. C. Yin, C. W. Wang, Optimality of the barrier strategy in de Finetti's dividend problem for spectrally negative Lévy processes: An alternative approach, *J. Comput. Appl. Math.* 233 (2009) 482-491